

# ANNULAR NONCROSSING PERMUTATIONS AND MINIMAL TRANSITIVE FACTORIZATIONS

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ABSTRACT. We give two combinatorial proofs of Goulden and Jackson's formula for the number of minimal transitive factorizations of a permutation when the permutation has two cycles. We use the recent result of Goulden, Nica, and Oancea on the number of maximal chains of annular noncrossing partitions of type  $B$ .

## 1. INTRODUCTION

Given an integer partition  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$  of  $n$ , denote by  $\alpha_\lambda$  the permutation

$$(1 \dots \lambda_1)(\lambda_1 + 1 \dots \lambda_1 + \lambda_2) \dots (n - \lambda_\ell + 1 \dots n)$$

of the set  $\{1, 2, \dots, n\}$  in the cycle notation. Let  $\mathcal{F}_\lambda$  be the set of all  $(n + \ell - 2)$ -tuples  $(\eta_1, \dots, \eta_{n+\ell-2})$  of transpositions such that

- (1)  $\eta_1 \cdots \eta_{n+\ell-2} = \alpha_\lambda$  and
- (2)  $\{\eta_1, \dots, \eta_{n+\ell-2}\}$  generates the symmetric group  $\mathcal{S}_n$ .

Such tuples are called *minimal transitive factorizations* of the permutation  $\alpha_\lambda$  of type  $\lambda$ , which are related to the branched covers of the sphere suggested by Hurwitz [Hur91, Str96].

In 1997, using algebraic methods Goulden and Jackson [GJ97] proved that

$$|\mathcal{F}_\lambda| = (n + \ell - 2)! n^{\ell-3} \prod_{i=1}^{\ell} \frac{\lambda_i^{\lambda_i}}{(\lambda_i - 1)!}. \quad (1)$$

Bousquet-Mélou and Schaeffer [BMS00] proved a more general formula than (1) and obtained (1) using the principle of inclusion and exclusion. Irving [Irv09] studied the enumeration of minimal transitive factorizations into cycles instead of transpositions.

If  $\lambda = (n)$ , the formula (1) yields

$$|\mathcal{F}_{(n)}| = n^{n-2}, \quad (2)$$

and there are several combinatorial proofs of (2) [Bia02, GY02, Mos89].

If  $\lambda = (p, q)$ , the formula (1) yields

$$|\mathcal{F}_{(p,q)}| = \frac{pq}{p+q} \binom{p+q}{q} p^p q^q. \quad (3)$$

A few special cases of (3) have bijective proofs: by Kim and Seo [KS03] for the case  $(p, q) = (1, n-1)$ , and by Rattan [Rat06] for the cases  $(p, q) = (2, n-2)$  and  $(p, q) = (3, n-3)$ . There are no simple combinatorial proofs for other  $(p, q)$ .

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Recently, Goulden et al. [GNO11] showed that the number of maximal chains in the poset  $NC^{(B)}(p, q)$  of annular noncrossing partitions of type  $B$  is

$$\binom{p+q}{q} p^p q^q + \sum_{c \geq 1} 2c \binom{p+q}{p-c} p^{p-c} q^{q+c}. \quad (4)$$

Interestingly it turns out that half the sum in (4) is equal to the number in (3):

$$\sum_{c \geq 1} c \binom{p+q}{p-c} p^{p-c} q^{q+c} = \frac{pq}{p+q} \binom{p+q}{q} p^p q^q.$$

In this paper we will give two combinatorial proofs of (3) using the results in [GNO11]. The rest of this paper is organized as follows. In Section 2 we recall the poset  $\mathcal{S}_{\text{nc}}^B(p, q)$  of annular noncrossing permutations of type  $B$  which is isomorphic to the poset  $NC^{(B)}(p, q)$  of annular noncrossing partitions of type  $B$ , and show that the number of connected maximal chains in  $\mathcal{S}_{\text{nc}}^B(p, q)$  is equal to  $\frac{2pq}{p+q} \binom{p+q}{q} p^p q^q$ . In Section 3 we prove that there is a 2-1 map from the set of connected maximal chains in  $\mathcal{S}_{\text{nc}}^B(p, q)$  to  $\mathcal{F}_{(p,q)}$ , thus completing a combinatorial proof of (3). In Section 4 we give another combinatorial proof of (3) by introducing marked annular noncrossing permutations of type  $A$ .

## 2. CONNECTED MAXIMAL CHAINS

A *signed permutation* is a permutation  $\sigma$  on  $\{\pm 1, \dots, \pm n\}$  satisfying  $\sigma(-i) = -\sigma(i)$  for all  $i \in \{1, \dots, n\}$ . We denote by  $B_n$  the set of signed permutations on  $\{\pm 1, \dots, \pm n\}$ .

We will use the two notations

$$\begin{aligned} [a_1 \ a_2 \ \dots \ a_k] &= (a_1 \ a_2 \ \dots \ a_k \ -a_1 \ -a_2 \ \dots \ -a_k), \\ ((a_1 \ a_2 \ \dots \ a_k)) &= (a_1 \ a_2 \ \dots \ a_k)(-a_1 \ -a_2 \ \dots \ -a_k), \end{aligned}$$

and call  $[a_1 \ a_2 \ \dots \ a_k]$  a *zero cycle* and  $((a_1 \ a_2 \ \dots \ a_k))$  a *paired nonzero cycle*. We also call the cycles  $\epsilon_i := [i] = (i \ -i)$  and  $((i \ j))$  *type B transpositions*, or simply transpositions if there is no possibility of confusion.

For  $\pi \in B_n$ , the *absolute length*  $\ell(\pi)$  is defined to be the smallest integer  $k$  such that  $\pi$  can be written as a product of  $k$  type  $B$  transpositions. The *absolute order* on  $B_n$  is defined by

$$\pi \leq \sigma \iff \ell(\sigma) = \ell(\pi) + \ell(\pi^{-1}\sigma).$$

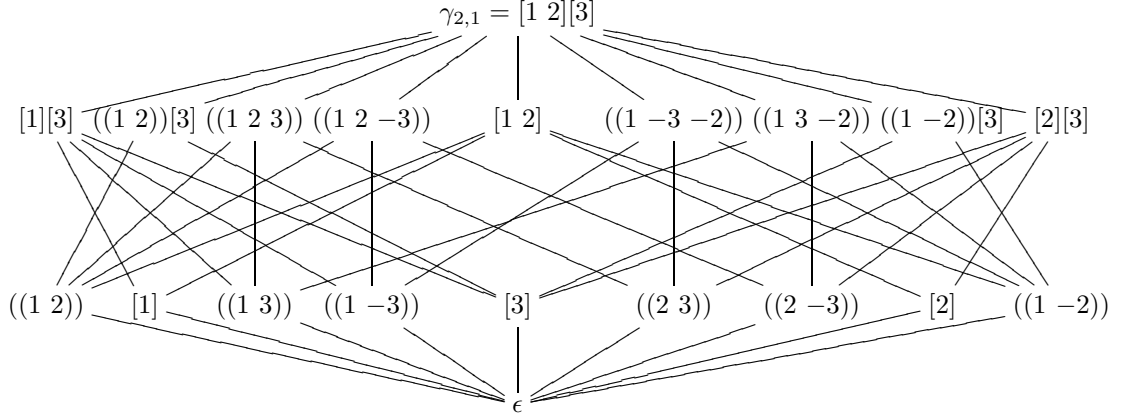
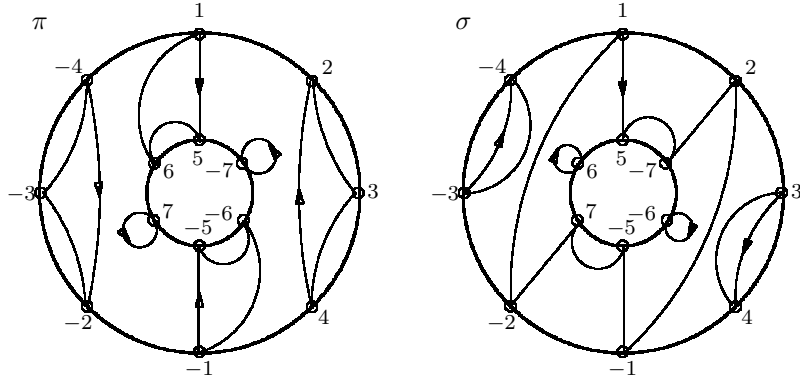
From now, we fix positive integers  $p$  and  $q$ . The poset  $\mathcal{S}_{\text{nc}}^B(p, q)$  of *annular noncrossing permutations of type B* is defined by

$$\mathcal{S}_{\text{nc}}^B(p, q) := [\epsilon, \gamma_{p,q}] = \{\sigma \in B_{p+q} : \epsilon \leq \sigma \leq \gamma_{p,q}\} \subseteq B_{p+q},$$

where  $\epsilon$  is the identity in  $B_{p+q}$  and  $\gamma_{p,q} = [1 \ \dots \ p][p+1 \ \dots \ p+q]$ . Figure 1 shows the Hasse diagram for  $\mathcal{S}_{\text{nc}}^B(2, 1)$ . Then  $\mathcal{S}_{\text{nc}}^B(p, q)$  is a graded poset with rank function

$$\text{rank}(\sigma) = (p+q) - (\# \text{ of paired nonzero cycles of } \sigma). \quad (5)$$

Nica and Oancea [NO09] showed that  $\sigma \in \mathcal{S}_{\text{nc}}^B(p, q)$  if and only if  $\sigma$  can be drawn without crossing arrows inside an annulus in which the outer circle has integers  $1, 2, \dots, p, -1, -2, \dots, -p$  in clockwise order and the inner circle has integers  $p+1, p+2, \dots, p+q, -p-1, -p-2, \dots, -p-q$  in counterclockwise order, see Figure 2. They also showed that  $\mathcal{S}_{\text{nc}}^B(p, q)$  is isomorphic to the poset  $NC^{(B)}(p, q)$  of annular noncrossing partitions of type  $B$ .


 FIGURE 1. The Hasse diagram for  $\mathcal{S}_{\text{nc}}^B(2, 1)$ .

 FIGURE 2.  $\pi = ((1 \ 5 \ 6))((2 \ 3 \ 4))$  and  $\sigma = [1 \ 5 \ -7 \ 2]((3 \ 4))$  in  $\mathcal{S}_{\text{nc}}^B(4, 3)$ 

A paired nonzero cycle  $((a_1 \ a_2 \ \dots \ a_k))$  is called *connected* if the set  $\{a_1, \dots, a_k\}$  intersects with both  $\{\pm 1, \dots, \pm p\}$  and  $\{\pm(p+1), \dots, \pm(p+q)\}$ , and *disconnected* otherwise. A zero cycle is always considered to be disconnected. For  $\sigma \in \mathcal{S}_{\text{nc}}^B(p, q)$ , the *connectivity* of  $\sigma$  is the number of connected paired nonzero cycles of  $\sigma$ .

We say that a maximal chain  $C = \{\epsilon = \pi_0 < \pi_1 < \dots < \pi_{p+q} = \gamma_{p,q}\}$  of  $\mathcal{S}_{\text{nc}}^B(p, q)$  is *disconnected* if the connectivity of each  $\pi_i$  is zero. Otherwise,  $C$  is called *connected*. Denote by  $\mathcal{CM}(\mathcal{S}_{\text{nc}}^B(p, q))$  the set of connected maximal chains of  $\mathcal{S}_{\text{nc}}^B(p, q)$ .

For a maximal chain  $C = \{\pi_0 < \pi_1 < \dots < \pi_n\}$  of the interval  $[\pi_0, \pi_n]$ , we define  $\varphi(C) = (\tau_1, \tau_2, \dots, \tau_n)$ , where  $\tau_i = \pi_i^{-1} \pi_{i+1}$ . Note that each  $\tau_i$  is a type  $B$  transposition and  $\pi_i = \tau_1 \tau_2 \dots \tau_i$  for all  $i = 1, 2, \dots, n$ .

**Lemma 1.** *If  $C$  is a connected maximal chain of  $\mathcal{S}_{\text{nc}}^B(p, q)$ , then  $\varphi(C)$  has no transpositions of the form  $\epsilon_i = [i]$  and has at least one connected transposition. If  $C$  is a disconnected maximal chain of  $\mathcal{S}_{\text{nc}}^B(p, q)$ , then  $\varphi(C)$  has only disconnected transpositions.*

*Proof.* By (5),  $\sigma$  covers  $\pi$  in  $\mathcal{S}_{\text{nc}}^B(p, q)$  if and only if one of the following conditions holds, see [NO09, Proposition 2.2]:

- (a)  $\pi^{-1}\sigma = \epsilon_i$  and the cycle containing  $i$  in  $\pi$  is nonzero, i.e.,  $\pi$  has  $((i \cdots))$  and  $\sigma$  has  $[i \cdots]$ .
- (b)  $\pi^{-1}\sigma = ((i \ j))$  and no two of  $i, -i, j, -j$  belong to the same cycle in  $\pi$  with  $|i| \neq |j|$ , i.e.,  $\pi$  has  $((i \cdots))(j \cdots)$  and  $\sigma$  has  $((i \cdots j \cdots))$ .
- (c)  $\pi^{-1}\sigma = ((i \ j))$  and the cycle containing  $i$  in  $\pi$  is nonzero and the cycle containing  $j$  in  $\pi$  is zero with  $|i| \neq |j|$ , i.e.,  $\pi$  has  $((i \cdots))[j \cdots]$  and  $\sigma$  has  $[i \cdots - j \cdots]$ .
- (d)  $\pi^{-1}\sigma = ((i \ j))$  and  $i$  and  $-j$  belong to the same nonzero cycle in  $\pi$  with  $|i| \neq |j|$ , i.e.,  $\pi$  has  $((i \cdots - j \cdots))$  and  $\sigma$  has  $[i \cdots][-j \cdots]$ .

If  $\sigma$  covers  $\pi$  in  $\mathcal{S}_{\text{nc}}^B(p, q)$ , we have  $\text{zc}(\sigma) \geq \text{zc}(\pi)$ , where  $\text{zc}(\sigma)$  is the the number of zero cycles in  $\sigma$ . More precisely we have

$$\text{zc}(\sigma) - \text{zc}(\pi) = \begin{cases} 0 & \text{if type (b) or (c),} \\ 1 & \text{if type (a),} \\ 2 & \text{if type (d).} \end{cases}$$

Since  $\gamma_{p,q}$  has two zero cycles, each  $\pi \in \mathcal{S}_{\text{nc}}^B(p, q)$  has at most two zero cycles. Moreover, if  $\pi$  has two zero cycles, then one of them belongs to  $\{\pm 1, \dots, \pm p\}$  and the other belongs to  $\{\pm(p+1), \dots, \pm(p+q)\}$ . Consider a maximal chain  $C$  in  $\mathcal{S}_{\text{nc}}^B(p, q)$ .

- If  $C$  has a permutation  $\pi$  with  $\text{zc}(\pi) = 1$ , there are two cover relations of type (a) and no cover relations of type (d) in  $C$ . For each cover relation  $\pi < \sigma$  of type (a), (b), or (c),  $\sigma$  is obtained by merging cycles in  $\pi$ . Since  $\gamma_{p,q}$  has only disconnected cycles, all permutations in  $C$  are disconnected, which implies that  $C$  is disconnected.
- Otherwise, there is a cover relation  $\pi < \sigma$  of type (d) in  $C$ . Then  $\sigma$  has two zero cycles  $[i \cdots]$  and  $[-j \cdots]$ , one of which is contained in  $\{\pm 1, \dots, \pm p\}$  and the other is contained in  $\{\pm(p+1), \dots, \pm(p+q)\}$ . Thus  $\pi$  has a connected nonzero cycle  $((i \cdots - j \cdots))$ , and  $C$  is connected. Since  $C$  has no cover relations of type (a),  $\varphi(C)$  has no transposition of the form  $\epsilon_i$ .

Therefore, if  $C$  is a disconnected maximal chain of  $\mathcal{S}_{\text{nc}}^B(p, q)$ , then  $\varphi(C)$  has two transpositions of the form  $\epsilon_i$ . So all transpositions of  $\varphi(C)$  are disconnected. Also, if  $C$  is a connected maximal chain of  $\mathcal{S}_{\text{nc}}^B(p, q)$ , then  $\varphi(C)$  has no transposition of the form  $\epsilon_i$  and has at least one connected transposition.  $\square$

The following proposition is a refinement of (4).

**Proposition 2.** *The number of disconnected maximal chains of  $\mathcal{S}_{\text{nc}}^B(p, q)$  is equal to*

$$\binom{p+q}{q} p^p q^q \quad (6)$$

*and the number of connected maximal chains of  $\mathcal{S}_{\text{nc}}^B(p, q)$  is equal to*

$$\sum_{c \geq 1} 2c \binom{p+q}{p-c} p^{p-c} q^{q+c}. \quad (7)$$

*Proof.* Let  $NC^{(B)}(n)$  denote the poset of *noncrossing partitions of type B* of size  $n$ . It is well-known [Rei97, Proposition 7] that the number of maximal chains of  $NC^{(B)}(n)$  equals  $n^n$ . Let  $\gamma_p := [1 \ 2 \ \dots \ p]$  and  $\gamma_q := [p+1 \ p+2 \ \dots \ p+q]$ . Since  $NC^{(B)}(p) \simeq [\epsilon, \gamma_p]$  and  $NC^{(B)}(q) \simeq [\epsilon, \gamma_q]$ , the numbers of maximal chains of  $[\epsilon, \gamma_p]$  and  $[\epsilon, \gamma_q]$  are respectively  $p^p$  and  $q^q$ . Given two maximal chains  $C_1$  of  $[\epsilon, \gamma_p]$  and  $C_2$  of  $[\epsilon, \gamma_q]$ , we can obtain a  $(p+q)$ -tuple  $(\tau_1, \dots, \tau_{p+q})$  of transpositions by shuffling  $\varphi(C_1)$  and  $\varphi(C_2)$  in  $\binom{p+q}{q}$  ways. Then

$C = \{\pi_0 < \pi_1 < \dots < \pi_{p+q}\}$ , where  $\pi_i = \tau_1 \dots \tau_i$ , is a disconnected maximal chain of  $\mathcal{S}_{\text{nc}}^B(p, q)$ . By Lemma 1, it is easy to see that every disconnected maximal chain of  $\mathcal{S}_{\text{nc}}^B(p, q)$  can be obtained in this way. Thus we get (6). By (4) and (6), we obtain (7).  $\square$

*Remark 1.* While one can also deduce Proposition 2 using the proof of Theorem 5.3 in [GNO11], our proof gives a direct combinatorial interpretation of (6).

We now prove the following identity that appears in the introduction. The proof is due to Krattenthaler [Kra].

**Lemma 3.** *We have*

$$\sum_{c \geq 1} c \binom{p+q}{p-c} p^{p-c} q^{q+c} = \frac{pq}{p+q} \binom{p+q}{q} p^p q^q. \quad (8)$$

*Proof.* Since  $c = p \cdot \frac{q+c}{p+q} - q \cdot \frac{p-c}{p+q}$ , we have

$$\begin{aligned} \sum_{c=0}^p c \binom{p+q}{p-c} p^{p-c} q^{q+c} &= \sum_{c=0}^p \left( p \cdot \frac{q+c}{p+q} - q \cdot \frac{p-c}{p+q} \right) \binom{p+q}{p-c} p^{p-c} q^{q+c} \\ &= \sum_{c=0}^p \left( \binom{p+q-1}{p-c} p^{p-c+1} q^{q+c} - \binom{p+q-1}{p-c-1} p^{p-c} q^{q+c+1} \right) \\ &= \binom{p+q-1}{p} p^{p+1} q^q = \frac{pq}{p+q} \binom{p+q}{p} p^p q^q. \end{aligned}$$

$\square$

By Proposition 2 and Lemma 3, we get the following.

**Corollary 4.** *The number of connected maximal chains of  $\mathcal{S}_{\text{nc}}^B(p, q)$  is equal to*

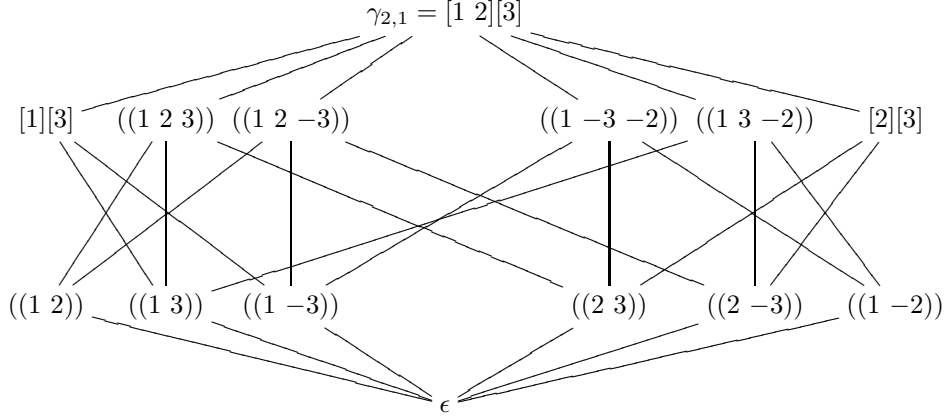
$$\frac{2pq}{p+q} \binom{p+q}{q} p^p q^q. \quad (9)$$

For example, Figure 3 illustrates  $16 = \frac{4}{3} \binom{3}{1} 2^2$  connected maximal chains of  $\mathcal{S}_{\text{nc}}^B(2, 1)$ .

By Corollary 4, in order to prove (3) combinatorially it is sufficient to find a 2-1 map from  $\mathcal{CM}(\mathcal{S}_{\text{nc}}^B(p, q))$  to  $\mathcal{F}_{(p,q)}$ . We will find such a map in the next section.

*Remark 2.* One can check that the factorizations  $\varphi(C)$  coming from connected maximal chains  $C$  in  $\mathcal{S}_{\text{nc}}^B(p, q)$  are precisely the minimal factorizations of  $\gamma_{p,q}$  in the Weyl group  $D_n$ . Thus Corollary 4 can be restated as follows: the number of minimal factorizations of  $\gamma_{p,q}$  in  $D_n$  is equal to  $\frac{2pq}{p+q} \binom{p+q}{q} p^p q^q$ . Goupil [Gou95, Theorem 3.1] also proved this result by finding a recurrence relation.

*Remark 3.* Since the proof of Lemma 3 is a simple manipulation, it is easy and straightforward to construct a combinatorial proof for the identity in Lemma 3. Together with the result in Section 3 we get a combinatorial proof of (9). It would be interesting to find a direct bijective proof of (9) without using Lemma 3.

FIGURE 3. Connected maximal chains in  $\mathcal{S}_{\text{nc}}^B(2, 1)$ .

### 3. A 2-1 MAP FROM $\mathcal{CM}(\mathcal{S}_{\text{nc}}^B(p, q))$ TO $\mathcal{F}_{(p,q)}$

Recall that a minimal transitive factorization of  $\alpha_{p,q} = (1 \dots p)(p+1 \dots p+q)$  is a sequence  $(\eta_1, \dots, \eta_{p+q})$  of transpositions in  $\mathcal{S}_{p+q}$  such that

- (1)  $\eta_1 \dots \eta_{p+q} = \alpha_{p,q}$  and
- (2)  $\{\eta_1, \dots, \eta_{p+q}\}$  generates  $\mathcal{S}_{p+q}$ ,

and  $\mathcal{F}_{(p,q)}$  is the set of minimal transitive factorizations of  $\alpha_{p,q}$ .

In this section we will prove the following theorem.

**Theorem 5.** *There is a 2-1 map from the set of connected maximal chains in  $\mathcal{S}_{\text{nc}}^B(p, q)$  to the set  $\mathcal{F}_{(p,q)}$  of minimal transitive factorizations of  $\alpha_{p,q}$ .*

In order to prove Theorem 5 we need some definitions.

**Definition 6** (Two maps  $(\cdot)^+$  and  $|\cdot|$ ). We introduce the following two maps.

- (1) The map  $(\cdot)^+ : B_n \rightarrow B_n$  is defined by

$$\sigma^+(i) = \begin{cases} |\sigma(i)| & \text{if } i > 0, \\ -|\sigma(i)| & \text{if } i < 0. \end{cases}$$

- (2) The map  $|\cdot| : B_n \rightarrow \mathcal{S}_n$  is defined by  $|\sigma|(i) = |\sigma(i)|$  for all  $i \in \{1, \dots, n\}$ .

**Definition 7.** A  $(p+q)$ -tuple  $(\tau_1, \dots, \tau_{p+q})$  of transpositions in  $B_{p+q}$  is called a *minimal transitive factorization of type B* of  $\gamma_{p,q} = [1 \dots p][p+1 \dots p+q]$  if it satisfies

- (1)  $\tau_1 \dots \tau_{p+q} = \gamma_{p,q}$ ,
- (2)  $\{|\tau_1|, \dots, |\tau_{p+q}|\}$  generates  $\mathcal{S}_{p+q}$ .

Denote by  $\mathcal{F}_{(p,q)}^{(B)}$  the set of minimal transitive factorizations of type B of  $\gamma_{p,q}$ .

**Definition 8.** A  $(p+q)$ -tuple  $(\sigma_1, \dots, \sigma_{p+q})$  of transpositions in  $B_{p+q}$  is called a *positive minimal transitive factorization of type B* of  $\beta_{p,q} = ((1 \dots p))((p+1 \dots p+q))$  if it satisfies

- (1)  $\sigma_1 \dots \sigma_{p+q} = \beta_{p,q}$ ,
- (2)  $\{|\sigma_1|, \dots, |\sigma_{p+q}|\}$  generates  $\mathcal{S}_{p+q}$ ,
- (3)  $\sigma_i = \sigma_i^+$  for all  $i = 1, \dots, p+q$ .

Denote by  $\mathcal{F}_{(p,q)}^+$  the set of positive minimal transitive factorizations of type  $B$  of  $\beta_{p,q}$ .

For the rest of this section we will prove the following:

- (1) The map  $\varphi : \mathcal{CM}(\mathcal{S}_{\text{nc}}^B(p, q)) \rightarrow \mathcal{F}_{(p,q)}^{(B)}$  is a bijection. (Lemma 9)
- (2) There is a 2-1 map  $(\cdot)^+ : \mathcal{F}_{(p,q)}^{(B)} \rightarrow \mathcal{F}_{(p,q)}^+$ . (Lemma 11)
- (3) There is a bijection  $|\cdot| : \mathcal{F}_{(p,q)}^+ \rightarrow \mathcal{F}_{(p,q)}$ . (Lemma 10)

By the above three statements the composition  $|\varphi^+| := |\cdot| \circ (\cdot)^+ \circ \varphi$  is a 2-1 map from  $\mathcal{CM}(\mathcal{S}_{\text{nc}}^B(p, q))$  to  $\mathcal{F}_{(p,q)}$ , which completes the proof of Theorem 5. Since the proofs of the first and the third statements are simpler, we will present these first.

**Lemma 9.** *The map  $\varphi : \mathcal{CM}(\mathcal{S}_{\text{nc}}^B(p, q)) \rightarrow \mathcal{F}_{(p,q)}^{(B)}$  is a bijection.*

*Proof.* Given a connected maximal chain  $C = \{\epsilon = \pi_0 < \pi_1 < \dots < \pi_{p+q} = \gamma_{p,q}\}$  in  $\mathcal{S}_{\text{nc}}^B(p, q)$ , the elements in the sequence  $\varphi(C) = (\tau_1, \dots, \tau_{p+q})$  are transpositions with  $\tau_1 \dots \tau_{p+q} = \gamma_{p,q}$ . By Lemma 1, at least one of  $\tau_i$ 's is connected. Thus  $\{|\tau_1|, \dots, |\tau_{p+q}|\}$  generates  $\mathcal{S}_{p+q}$ , and  $\varphi(C) \in \mathcal{F}_{(p,q)}^{(B)}$ . Conversely, if  $\tau = (\tau_1, \dots, \tau_{p+q}) \in \mathcal{F}_{(p,q)}^{(B)}$ , then  $\varphi^{-1}(\tau) = \{\epsilon = \pi_0 < \pi_1 < \dots < \pi_{p+q} = \gamma_{p,q}\}$ , where  $\pi_i = \tau_1 \dots \tau_i$ , is a connected maximal chain in  $\mathcal{S}_{\text{nc}}^B(p, q)$  because  $\{|\tau_1|, \dots, |\tau_{p+q}|\}$  generates  $\mathcal{S}_{p+q}$ .  $\square$

**Lemma 10.** *There is a bijection  $|\cdot| : \mathcal{F}_{(p,q)}^+ \rightarrow \mathcal{F}_{(p,q)}$ .*

*Proof.* Let  $(\sigma_1, \dots, \sigma_{p+q}) \in \mathcal{F}_{(p,q)}^+$ . Each  $\sigma_i$  can be written as  $\sigma_i = ((j \ k))$  for some positive integers  $j$  and  $k$ . In this case we let  $\eta_i = |\sigma_i| = (j \ k) \in \mathcal{S}_{p+q}$ . Then the map  $|\cdot| : \mathcal{F}_{(p,q)}^+ \rightarrow \mathcal{F}_{(p,q)}$  sending  $(\sigma_1, \dots, \sigma_{p+q})$  to  $(\eta_1, \dots, \eta_{p+q})$  is a bijection.  $\square$

Recall  $\epsilon_i = [i] = (i \ -i)$ . We write  $\overline{((i \ j))} := ((i \ -j))$ . It is easy to see that for  $i, j \in \{\pm 1, \dots, \pm(p+q)\}$ , we have

$$[i \ j] = \epsilon_i((i \ j)) = ((i \ j))\epsilon_j = \overline{((i \ j))}\epsilon_i = \epsilon_j\overline{((i \ j))}. \quad (10)$$

**Lemma 11.** *There is a 2-1 map  $(\cdot)^+ : \mathcal{F}_{(p,q)}^{(B)} \rightarrow \mathcal{F}_{(p,q)}^+$ .*

*Proof.* For  $(\tau_1, \tau_2, \dots, \tau_{p+q}) \in \mathcal{F}_{(p,q)}^{(B)}$ , we define  $(\tau_1, \tau_2, \dots, \tau_{p+q})^+ = (\tau_1^+, \tau_2^+, \dots, \tau_{p+q}^+)$ . Since  $\tau_1^+ \dots \tau_{p+q}^+ = \gamma_{p,q}^+ = \beta_{p,q}$ , we have  $(\tau_1, \tau_2, \dots, \tau_{p+q})^+ \in \mathcal{F}_{(p,q)}^+$ .

The map  $(\cdot)^+$  is surjective: Suppose  $(\sigma_1, \sigma_2, \dots, \sigma_{p+q}) \in \mathcal{F}_{(p,q)}^+$ . Since  $\sigma_1 \sigma_2 \dots \sigma_{p+q} = \beta_{p,q}$  and  $\gamma_{p,q} = \epsilon_{p+1} \epsilon_1 \beta_{p,q}$ , we have

$$\gamma_{p,q} = \epsilon_{p+1} \epsilon_1 \sigma_1 \sigma_2 \dots \sigma_{p+q}. \quad (11)$$

By (10), if  $\sigma_\ell = ((u \ v))$ , we have

$$\epsilon_u \sigma_1 \sigma_2 \dots \sigma_{p+q} = \epsilon_v \tilde{\sigma}_1 \dots \tilde{\sigma}_{\ell-1} \overline{\sigma_\ell} \sigma_{\ell+1} \dots \sigma_{p+q}, \quad (12)$$

where

$$\tilde{\sigma}_i = \begin{cases} \overline{\sigma_i} & \text{if } \sigma_i \text{ has either } u \text{ or } v, \\ \sigma_i & \text{otherwise.} \end{cases}$$

Since  $(\sigma_1, \sigma_2, \dots, \sigma_{p+q})$  is transitive, we can find integers  $1 = a_0, a_1, \dots, a_k = p+1$  such that  $((a_{i-1}, a_i)) \in \{\sigma_1, \sigma_2, \dots, \sigma_{p+q}\}$  for all  $1 \leq i \leq k$ . Using this fact and the relation in (12), we can rewrite (11) as

$$\gamma_{p,q} = \epsilon_{p+1}(\epsilon_{p+1}\tau_1\tau_2 \dots \tau_{p+q}) = \tau_1\tau_2 \dots \tau_{p+q},$$

where  $\tau_i = \sigma_i$  or  $\overline{\sigma_i}$  for all  $i = 1, 2, \dots, p+q$ . Hence,  $(\tau_1, \tau_2, \dots, \tau_{p+q})^+ = (\sigma_1, \sigma_2, \dots, \sigma_{p+q})$  and  $(\cdot)^+$  is surjective.

We need to show that  $(\cdot)^+$  is two-to-one. Let us fix  $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_{p+q}) \in \mathcal{F}_{(p,q)}^+$ . Since  $(\cdot)^+$  is surjective, there is  $\tau = (\tau_1, \dots, \tau_{p+q}) \in \mathcal{F}_{(p,q)}^{(B)}$  satisfying  $\tau^+ = \sigma$ . Then  $\tau' = (\tau'_1 \dots \tau'_{p+q}) \in \mathcal{F}_{(p,q)}^{(B)}$  defined by

$$\tau'_i = \begin{cases} \tau_i & \text{if } \tau_i \text{ is disconnected} \\ \overline{\tau_i} & \text{if } \tau_i \text{ is connected,} \end{cases} \quad (13)$$

also satisfies  $(\tau')^+ = \sigma$ . Since  $\tau$  has at least one connected transposition,  $\tau \neq \tau'$ . Hence the preimage of  $\sigma$  under  $(\cdot)^+$  has at least two elements. In order to prove that the preimage of  $\sigma$  under  $(\cdot)^+$  has exactly two elements, we consider the sets

$$\begin{aligned} \mathcal{F} &= \{(\tau_1, \dots, \tau_{p+q}) : \tau_i^+ = \sigma_i \text{ for all } i = 1, \dots, p+q\}, \\ \mathcal{F}(\delta) &= \{(\tau_1, \dots, \tau_{p+q}) : \tau_1 \dots \tau_{p+q} = \delta, \tau_i^+ = \sigma_i \text{ for all } i = 1, \dots, p+q\}. \end{aligned}$$

Then  $\mathcal{F}$  has  $2^{p+q}$  elements, and all preimages of  $\sigma$  under  $(\cdot)^+$  belong to  $\mathcal{F}(\gamma_{p,q})$ .

Suppose  $(\tau_1, \dots, \tau_{p+q}) \in \mathcal{F}(\delta)$ . Since  $\delta^+ = \beta_{p,q}$ , we have  $(\delta \beta_{p,q}^{-1})^+ = \epsilon$ . Since  $\delta \beta_{p,q}^{-1}$  is an even permutation as a permutation on  $\{\pm 1, \dots, \pm(p+q)\}$ , we have  $\delta \beta_{p,q}^{-1} = \epsilon_{i_1} \dots \epsilon_{i_{2k}}$  for some  $1 \leq i_1 < \dots < i_{2k} \leq p+q$ . Thus the set

$$I(\delta) := \{i : \delta(i) = -\beta_{p,q}(i) \text{ and } 1 \leq i \leq p+q\}$$

has even cardinality. Define the set

$$B(\beta_{p,q}) = \{\delta \in B_{p+q} : \delta^+ = \beta_{p,q} \text{ and } \#I(\delta) \text{ is even}\},$$

whose cardinality is  $2^{p+q-1}$ . Then we have

$$\#\mathcal{F} = \sum_{\delta \in B(\beta_{p,q})} \#\mathcal{F}(\delta). \quad (14)$$

We claim that  $\#\mathcal{F}(\delta) \geq 2$  for each  $\delta \in B(\beta_{p,q})$ . Then by the claim together with  $\#\mathcal{F} = 2^{p+q}$ ,  $\#\mathcal{F}(\delta) = 2^{p+q-1}$ , and (14), we get  $\#\mathcal{F}(\delta) = 2$  for each  $\delta \in B(\beta_{p,q})$ . In particular, we have  $\#\mathcal{F}(\gamma_{p,q}) = 2$ , which implies that the preimage of  $\sigma$  under  $(\cdot)^+$  has exactly two elements, thus completing the proof of this lemma.

It remains to show the claim. Suppose  $\delta \in B(\beta_{p,q})$ . Then we have

$$\delta = \left( \prod_{i \in I(\delta)} \epsilon_i \right) \sigma_1 \sigma_2 \dots \sigma_{p+q}. \quad (15)$$

Using the relation (12) and the transitivity, we can rewrite (15) as

$$\delta = \epsilon_1^{\#I(\delta)} \tau_1 \tau_2 \dots \tau_{p+q} = \tau_1 \tau_2 \dots \tau_{p+q},$$

where  $\tau_i = \sigma_i$  or  $\overline{\sigma_i}$  for all  $i = 1, 2, \dots, p+q$ . Then  $\mathcal{F}(\delta)$  has at least two elements  $(\tau_1, \dots, \tau_n)$  and  $(\tau'_1, \dots, \tau'_n)$ , the latter is defined by (13). Thus  $\#\mathcal{F}(\delta) \geq 2$  and we are done.  $\square$



For example, let  $\sigma = ( ((1\ 2)), ((2\ 5)), ((2\ 3)), ((4\ 5)), ((3\ 4)) ) \in \mathcal{F}_{(3,2)}^+$  be the following factorization

$$\beta_{3,2} = ((1\ 2\ 3))((4\ 5)) = ((1\ 2)) ((2\ 5)) ((2\ 3)) ((4\ 5)) ((3\ 4)).$$

Since  $\gamma_{3,2} = \epsilon_4 \epsilon_1 \beta_{3,2}$ , we can obtain a factorization of  $\gamma_{3,2}$  from  $\sigma$  as follows:

$$\begin{aligned} \gamma_{3,2} &= [1\ 2\ 3][4\ 5] = \epsilon_4 \epsilon_1 ((1\ 2)) ((2\ 5)) ((2\ 3)) ((4\ 5)) ((3\ 4)) \\ &= \epsilon_4 \epsilon_2 \overline{((1\ 2))} ((2\ 5)) ((2\ 3)) ((4\ 5)) ((3\ 4)) \\ &= \epsilon_4 \epsilon_3 ((1\ 2)) \overline{((2\ 5))} \overline{((2\ 3))} ((4\ 5)) ((3\ 4)) \\ &= \epsilon_4 \epsilon_4 ((1\ 2)) \overline{((2\ 5))} ((2\ 3)) \overline{((4\ 5))} \overline{((3\ 4))} \\ &= ((1\ 2)) \overline{((2\ 5))} ((2\ 3)) \overline{((4\ 5))} \overline{((3\ 4))}. \end{aligned}$$

Thus  $\tau = ( ((1\ 2)), \overline{((2\ 5))}, ((2\ 3)), \overline{((4\ 5))}, \overline{((3\ 4))} ) \in \mathcal{F}_{(3,2)}^{(B)}$  satisfies  $\tau^+ = \sigma$ . The factorization  $\tau' = ( ((1\ 2)), ((2\ 5)), ((2\ 3)), \overline{((4\ 5))}, ((3\ 4)) )$  obtained by toggling the connected transpositions of  $\tau$  also satisfies  $(\tau')^+ = \sigma$ .

#### 4. MARKED ANNULAR NONCROSSING PERMUTATIONS OF TYPE A

Mingo and Nica [MN04] studied the set  $\mathcal{S}_{\text{nc}}^A(p, q) = \{\pi \in \mathcal{S}_n : \pi \leq \alpha_{p,q}\}$  of annular noncrossing permutations of type A. In contrast to the type B case,  $\mathcal{S}_{\text{nc}}^A(p, q)$  is not isomorphic to the set  $NC^A(p, q)$  of annular noncrossing partitions of type A. In fact, the two sets  $\mathcal{S}_{\text{nc}}^A(p, q)$  and  $NC^A(p, q)$  have different cardinalities, see [MN04, Section 4].

In what follows we construct a poset whose maximal chains are in bijection with minimal transitive factorizations of  $\alpha_{p,q}$ .

Recall that the (*absolute*) *length*  $\ell(\pi)$  of  $\pi \in \mathcal{S}_n$  is defined to be the smallest integer  $k$  such that  $\pi$  can be written as a product of  $k$  transpositions. Equivalently,  $\ell(\pi) = n - \text{cycle}(\pi)$ , where  $\text{cycle}(\pi)$  is the number of cycles in  $\pi$ . The (*absolute*) *order*  $\pi \leq \sigma$  is defined if and only if  $\ell(\sigma) = \ell(\pi) + \ell(\pi^{-1}\sigma)$ . In this order, the interval  $[\epsilon, \alpha_{p,q}]$  is isomorphic to  $[\epsilon, (1, 2, \dots, p)] \times [\epsilon, (p+1, p+2, \dots, p+q)]$ .

Similarly to the type B case, we say that  $\pi \in \mathcal{S}_{p+q}$  is *connected* if  $\pi$  has a cycle intersecting with both  $\{1, 2, \dots, p\}$  and  $\{p+1, p+2, \dots, p+q\}$ , and *disconnected* otherwise.

A *marked annular noncrossing permutation of type A* is a pair  $(\pi, z)$  of a permutation  $\pi \in \mathcal{S}_{p+q}$  and an integer  $z \in \{0, 1\}$  such that

- (1) if  $\pi$  is disconnected, then  $\pi \leq \alpha_{p,q}$ , i.e.  $\ell(\alpha_{p,q}) = \ell(\pi) + \ell(\pi^{-1}\alpha_{p,q})$ ,
- (2) if  $\pi$  is connected, then  $\ell(\alpha_{p,q}) = \ell(\pi) + \ell(\pi^{-1}\alpha_{p,q}) - 2$ ,
- (3) if  $z = 1$ , then  $\pi$  is disconnected.

We denote by  $\overline{\mathcal{S}}_{\text{nc}}^A(p, q)$  the set of marked annular noncrossing permutations of type A. We define the partial order  $\leq$  on  $\overline{\mathcal{S}}_{\text{nc}}^A(p, q)$  as follows:  $(\pi, z) \leq (\sigma, w)$  if and only if one of the following holds:

- (1)  $z = w$  and  $\pi \leq \sigma$ ,
- (2)  $z = 0, w = 1$ ,  $\pi$  is connected,  $\sigma$  is disconnected, and  $\ell(\sigma) = \ell(\pi) + \ell(\pi^{-1}\sigma) - 2$ .

Then  $\overline{\mathcal{S}}_{\text{nc}}^A(p, q)$  is a graded poset of rank  $p + q$  with minimum  $\hat{0} = (\epsilon, 0)$  and maximum  $\hat{1} = (\alpha_{p,q}, 1)$ . The rank function of  $\overline{\mathcal{S}}_{\text{nc}}^A(p, q)$  is given by

$$\text{rank}(\pi, z) = \ell(\pi) + 2z.$$

We say that a multichain  $(\pi_1, z_1) \leq (\pi_2, z_2) \leq \dots \leq (\pi_m, z_m)$  of  $\overline{\mathcal{S}}_{\text{nc}}^A(p, q)$  is *connected* if it contains at least one connected permutation, and *disconnected* otherwise.

We now show the relation between the maximal chains of  $\overline{\mathcal{S}}_{\text{nc}}^A(p, q)$  and the minimal transitive factorizations of  $\alpha_{p,q}$ .

**Proposition 12.** *There is a bijection between the set of maximal chains of  $\overline{\mathcal{S}}_{\text{nc}}^A(p, q)$  and the set of minimal transitive factorizations of  $\alpha_{p,q}$ . Moreover, every maximal chain of  $\overline{\mathcal{S}}_{\text{nc}}^A(p, q)$  is connected.*

*Proof.* Let  $(\epsilon, 0) = (\pi_0, z_0) < (\pi_1, z_1) < \dots < (\pi_{p+q}, z_{p+q}) = (\alpha_{p,q}, 1)$  be a maximal chain in  $\overline{\mathcal{S}}_{\text{nc}}^A(p, q)$ . By definition of the partial order on  $\overline{\mathcal{S}}_{\text{nc}}^A(p, q)$  there is a unique integer  $k$  such that  $z_0 = z_1 = \dots = z_{k-1} = 0$  and  $z_k = z_{k+1} = \dots = z_{p+q} = 1$ .

Suppose  $i \in \{1, 2, \dots, p+q\} \setminus \{k\}$ . We have  $\ell(\pi_i) = \ell(\pi_{i-1}) + \ell(\pi_{i-1}^{-1}\pi_i)$ . Since  $z_i = z_{i-1}$  and

$$\ell(\pi_i) + 2z_i = \text{rank}(\pi_i, z_i) = \text{rank}(\pi_{i-1}, z_{i-1}) + 1 = \ell(\pi_{i-1}) + 2z_{i-1} + 1,$$

we get  $\ell(\pi_i) = \ell(\pi_{i-1}) + 1$ . Thus  $\ell(\pi_{i-1}^{-1}\pi_i) = 1$  and  $t_i = \pi_{i-1}^{-1}\pi_i$  is a transposition. Furthermore, since  $\pi_k, \pi_{k+1}, \dots, \pi_{p+q}$  are disconnected, so are  $t_{k+1}, t_{k+2}, \dots, t_{p+q}$ .

On the other hand, we have  $\ell(\pi_k) = \ell(\pi_{k-1}) + \ell(\pi_{k-1}^{-1}\pi_k) - 2$  and  $\ell(\pi_k) + 2 = \text{rank}(\pi_k, z_k) = \text{rank}(\pi_{k-1}, z_{k-1}) + 1 = \ell(\pi_{k-1}) + 1$ . Thus  $\ell(\pi_{k-1}^{-1}\pi_k) = 1$  and  $t_k = \pi_{k-1}^{-1}\pi_k$  is a connected transposition. In particular,  $t_k$  is the last connected transposition in  $t_1, t_2, \dots, t_{p+q}$ . It is easy to see that the map sending the maximal chain to  $(t_1, t_2, \dots, t_{p+q})$  is a desired bijection.  $\square$

**Proposition 13.** [GNO11, Proposition 5.1] *For positive integers  $p, q, m$ , there is a bijection between the set of tuples  $(c, d; L^E, R_1^E, \dots, R_m^E; L^I, R_1^I, \dots, R_m^I)$  satisfying  $c \geq 1$ ,  $1 \leq d \leq 2c$ , and*

$$L^E, R_1^E, \dots, R_m^E \subseteq \{1, 2, \dots, p\}, \quad |L^E| = |R_1^E| + \dots + |R_m^E| + c, \quad (16)$$

$$L^I, R_1^I, \dots, R_m^I \subseteq \{p+1, p+2, \dots, p+q\}, \quad |L^I| = |R_1^I| + \dots + |R_m^I| - c, \quad (17)$$

*and the set of connected multichains  $\pi_1 \leq \pi_2 \leq \dots \leq \pi_m$  in  $\mathcal{S}_{\text{nc}}^B(p, q)$  such that*

$$\text{rank}(\pi_i) = p + q - (|R_i^E| + \dots + |R_m^E| + |R_i^I| + \dots + |R_m^I|), \quad 1 \leq i \leq m.$$

We now prove a type  $A$  analog of Proposition 13. Our proof for the type  $A$  analog is almost the same as the proof of Proposition 5.1 in [GNO11]. The only difference (except the obvious difference caused by sign) is that for the type  $A$  case we have to determine the first and the last elements of a cycle. More precisely, for a cycle of the form  $((a_1, \dots, a_k))$  in  $\mathcal{S}_{\text{nc}}^B(p, q)$  whose elements are contained in  $\{\pm 1, \pm 2, \dots, \pm p\}$ , there is a unique way to write the cycle as  $((b_1, \dots, b_k))$  such that  $b_1, \dots, b_k, -b_1, \dots, -b_k$  are in the same cyclic order as the subsequence  $1, 2, \dots, p, -1, -2, \dots, -p$  consisting of  $\pm a_1, \pm a_2, \dots, \pm a_k$ . Thus we can naturally say that  $b_1$  (or  $-b_1$ ) is the first element and  $b_k$  (or  $-b_k$ ) is the last element of the cycle. For instance, consider the cycle  $((1, 2, -4))$  in, say,  $\mathcal{S}_{\text{nc}}^B(4, 3)$ . Then  $((-4, 1, 2))$  is the only way so that  $-4, 1, 2, 4, -1, -2$  are in the same cyclic order as the subsequence  $1, 2, 4, -1, -2, -4$  of  $1, 2, 3, 4, -1, -2, -3, -4$ . If we write the cycle as  $((1, 2, -4))$ , the sequence  $1, 2, -4, -1, -2, 4$  is not in the same cyclic order as  $1, 2, 4, -1, -2, -4$ . However, for the cycle  $(1, 2, 4)$  in  $\overline{\mathcal{S}}_{\text{nc}}^A(4, 3)$ , all three cyclic shifts of  $1, 2, 4$  are, of course, in the same cyclic order as the subsequence  $1, 2, 4$  of  $1, 2, 3, 4$ .

**Proposition 14.** *For positive integers  $p, q$ , and  $m$ , there is a bijection between the set of tuples  $(c, d; L^E, R_1^E, \dots, R_m^E; L^I, R_1^I, \dots, R_m^I)$  satisfying  $c \geq 1$ ,  $1 \leq d \leq c$ , (16), and (17), and the set of connected multichains  $(\pi_1, z_1) \leq (\pi_2, z_2) \leq \dots \leq (\pi_m, z_m)$  in  $\overline{\mathcal{S}}_{\text{nc}}^A(p, q)$  such that*

$$\text{rank}(\pi_i, z_i) = p + q - (|R_i^E| + \dots + |R_m^E| + |R_i^I| + \dots + |R_m^I|), \quad 1 \leq i \leq m. \quad (18)$$

*Proof.* Consider a connected multichain  $(\pi_1, z_1) \leq (\pi_2, z_2) \leq \dots \leq (\pi_m, z_m)$  in  $\overline{\mathcal{S}}_{\text{nc}}^A(p, q)$ . We will define the corresponding tuple  $(c, d; L^E, R_1^E, \dots, R_m^E; L^I, R_1^I, \dots, R_m^I)$  according to the following steps.

*Step 1.* We first determine two integers  $a \in \{1, 2, \dots, p\}$  and  $b \in \{p+1, \dots, p+q\}$ . Let  $k$  be the largest index such that  $\pi_k$  is connected. Then  $\pi_k$  has one or more connected cycles. Take the connected cycle  $C_{\max}$  of  $\pi_k$  with largest element. Since  $C_{\max}$  is connected, it can be uniquely written as  $C_{\max} = (a_1, \dots, a_r, b_1, \dots, b_s)$ , where  $1 \leq a_1, \dots, a_r \leq p$  and  $p+1 \leq b_1, \dots, b_s \leq p+q$ . We let  $a = a_1$  and  $b = b_s$ .

*Step 2.* To each  $(\pi_i, z_i)$  we associate a tuple  $(L_i^E, R_i^E; L_i^I, R_i^I)$  as follows. First, we set  $L_i^E = R_i^E = L_i^I = R_i^I = \emptyset$ , and for every cycle  $C$  of  $(\pi_i, z_i)$  we do the following. If  $C$  is contained in  $\{1, 2, \dots, p\}$ , add to  $L_i^E$  (resp.  $R_i^E$ ) the element of  $C$  that appears first (resp. last) in the sequence  $a, a+1, \dots, p, 1, 2, \dots, a-1$ . If  $C$  is contained in  $\{p+1, p+2, \dots, p+q\}$ , add to  $L_i^I$  (resp.  $R_i^I$ ) the element of  $C$  that appears first (resp. last) in the sequence  $b+1, b+2, \dots, p+q, p+1, p+2, \dots, b$ . If  $C$  is a connected cycle, it can be uniquely written as  $C = (g_1, \dots, g_u, h_1, \dots, h_v)$ , where  $1 \leq g_1, \dots, g_u \leq p$  and  $p+1 \leq h_1, \dots, h_v \leq p+q$ . In this case we add  $g_1$  to  $L_i^E$  and  $h_v$  to  $R_i^I$ .

*Step 3.* Let  $L^E = L_1^E \cup \dots \cup L_m^E$  and  $L^I = L_1^I \cup \dots \cup L_m^I$ . Now consider the sequence

$$a, a+1, \dots, p, 1, 2, \dots, a-1, b+1, b+2, \dots, p+q, p+1, p+2, \dots, b \quad (19)$$

with parenthesization obtained by placing a left parenthesis before every integer in  $L^E \cup L^I$ , a right parenthesis labeled  $i$  after every integer in  $R_i^E \cup R_i^I$  for all  $i = 1, 2, \dots, m$ . There may be more than one right parenthesis after one integer. In this case the right parentheses are placed in the increasing order of their labels. By the construction it is clear that the parenthesization is balanced. Now remove the integers larger than  $p$  and their left and right parentheses in (19). Then we have more left parentheses than right parentheses. Let  $c$  be the number of left parentheses minus the number of right parentheses. Then there are exactly  $c$  unmatched left parentheses. Let  $j_1 < j_2 < \dots < j_c$  be the integers whose left parentheses are unmatched. Note that the left parenthesis of  $a$  is unmatched because it was matched with a right parenthesis of  $b$  before removing the numbers with parentheses. We define  $d$  to be the index with  $j_d = a$ . We clearly have  $1 \leq d \leq c$ .

Then the map sending the multichain to  $(c, d; L^E, R_1^E, \dots, R_m^E; L^I, R_1^I, \dots, R_m^I)$  is a desired bijection. The inverse map can be obtained in the same way as in the proof Proposition 5.1 in [GNO11].  $\square$

**Corollary 15.** *The number of maximal chains in  $\overline{\mathcal{S}}_{\text{nc}}^A(p, q)$  is equal to*

$$\sum_{c \geq 1} c \binom{p+q}{p-c} p^{p-c} q^{q+c}.$$

*Proof.* Let  $M = \{(\pi_0, z_0) < (\pi_1, z_1) < \dots < (\pi_{p+q}, z_{p+q})\}$  be a maximal chain in  $\overline{\mathcal{S}}_{\text{nc}}^A(p, q)$ . By Proposition 12,  $M$  is connected. Let

$$(c, d; L^E, R_0^E, R_1^E, \dots, R_{p+q}^E; L^I, R_0^I, R_1^I, \dots, R_{p+q}^I) \quad (20)$$

be the tuple corresponding to  $M$  under the map in Proposition 14. Since  $M$  is maximal, using the conditions (16), (17), and (18), we have

$$\begin{aligned} |R_i^E| + |R_i^I| &= 1, \quad i = 0, 1, \dots, p+q-1, \\ R_{p+q}^E = R_{p+q}^I &= \emptyset, \quad L^E = \{1, 2, \dots, p\}, \quad L^I = \{p+1, p+2, \dots, p+q\}, \\ |R_0^E| + \dots + |R_{p+q-1}^E| &= p-c, \quad |R_0^I| + \dots + |R_{p+q-1}^I| = q+c, \end{aligned}$$

It is now easy to see that for fixed  $c$ , the number of choices for the tuples (20) satisfying the above conditions is  $\binom{p+q}{p-c} p^{p-c} q^{q+c}$ .  $\square$

*Remark 4.* Using Propositions 13 and 14 we obtain a 2-1 map between the sets of connected multichains in  $\mathcal{S}_{\text{nc}}^B(p, q)$  and in  $\overline{\mathcal{S}}_{\text{nc}}^A(p, q)$ . It is not difficult to check that this 2-1 map is essentially the same as the 2-1 map in the previous section. Note also that by Propositions 13 and 14 there is a 2-1 map from the set of connected permutations in  $\mathcal{S}_{\text{nc}}^B(p, q)$  to the set of connected permutations in  $\overline{\mathcal{S}}_{\text{nc}}^A(p, q)$ . However, we have  $|\mathcal{S}_{\text{nc}}^B(p, q)| \neq 2|\overline{\mathcal{S}}_{\text{nc}}^A(p, q)|$  because the numbers of disconnected permutations in  $\mathcal{S}_{\text{nc}}^B(p, q)$  and in  $\overline{\mathcal{S}}_{\text{nc}}^A(p, q)$  are equal to  $\binom{2p}{p}\binom{2q}{q}$  and  $2C_p C_q$  respectively, where  $C_n = \frac{1}{n+1}\binom{2n}{n}$ .

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#### REFERENCES

- [Bia02] P. Biane. Parking functions of types A and B. *Electron. J. Combin.*, 9(1):Note 7, 5 pp. (electronic), 2002.
- [BMS00] Mireille Bousquet-Mélou and Gilles Schaeffer. Enumeration of planar constellations. *Advances in Applied Mathematics*, 24:337–368, 2000.
- [GJ97] I. P. Goulden and D. M. Jackson. Transitive factorisations into transpositions and holomorphic mappings on the sphere. *Proc. Amer. Math. Soc.*, 125(1):51–60, 1997.
- [GNO11] I. Goulden, Alexandru Nica, and Ion Oancea. Enumerative properties of  $NC^{(B)}(p, q)$ . *Annals of Combinatorics*, 15:277–303, 2011. 10.1007/s00026-011-0095-4.
- [Gou95] Alain Goupil. Reflection decompositions in the classical Weyl groups. *Discrete Math.*, 137(1-3):195–209, 1995.
- [GY02] Ian Goulden and Alexander Yong. Tree-like properties of cycle factorizations. *J. Combin. Theory Ser. A*, 98(1):106–117, 2002.
- [Hur91] A. Hurwitz. Ueber Riemann'sche Flächen mit gegebenen Verzweigungspunkten. *Math. Ann.*, 39(1):1–60, 1891.
- [Irv09] John Irving. Minimal transitive factorizations of permutations into cycles. *Canad. J. Math.*, 61(5):1092–1117, 2009.
- [Kra] Christian Krattenthaler. Personal communication.
- [KS03] Dongsu Kim and Seunghyun Seo. Transitive cycle factorizations and prime parking functions. *J. Combin. Theory Ser. A*, 104(1):125–135, 2003.
- [MN04] James A. Mingo and Alexandru Nica. Annular noncrossing permutations and partitions, and second-order asymptotics for random matrices. *Int. Math. Res. Not.*, (28):1413–1460, 2004.
- [Mos89] Paul Moszkowski. A solution to a problem of Dénes: a bijection between trees and factorizations of cyclic permutations. *European J. Combin.*, 10(1):13–16, 1989.

- [NO09] Alexandru Nica and Ion Oancea. Posets of annular non-crossing partitions of types  $B$  and  $D$ . *Discrete Math.*, 309(6):1443–1466, 2009.
- [Rat06] Amarpreet Rattan. Permutation factorizations and prime parking functions. *Ann. Comb.*, 10(2):237–254, 2006.
- [Rei97] Victor Reiner. Non-crossing partitions for classical reflection groups. *Discrete Math.*, 177(1-3):195–222, 1997.
- [Str96] Volker Strehl. Minimal transitive products of transpositions—the reconstruction of a proof of A. Hurwitz. *Sém. Lothar. Combin.*, 37:Art. S37c, 12 pp. (electronic), 1996.

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